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Scaling and exotic regimes in decaying Burgers turbulence

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Abstract. We analyse the stochastic scaling laws arising in the invicid limit of the decaying solutions of the Burgers equation. The linear scaling of the velocity structure functions is shown to reflect the domination by shocks of the long-time asymptotics. We exhibit new self-similar statistics of solutions describing phases with diluted shocks. Some speculations are included on the nature of systems whose large time behaviour is described by the new statistics.

1. Introduction

The Burgers equation, a version of the Navier–Stokes equation without pressure, takes in the $1 + 1$ -dimensions the form

$$\partial_t u + u \partial_x u - \nu \partial_x^2 u = 0 \quad (1)$$

where $u = u(x, t)$ is the velocity field. We have not included the force term since we shall be interested in the free decay of initial data. Although we shall stick to the one-dimensional space, most of the following can be generalized to higher dimensions. We are interested in statistical properties of the velocity field at time $t > 0$, given the statistics of random initial data. Equation (1) has the $u(t, x) \mapsto -u(t, -x)$ symmetry which, if respected by the statistics of the initial conditions, will persist at all times. The problem is to evaluate the n -point correlation functions $\langle \prod_j u(x_j, t) \rangle$ at equal time in the invicid limit $\nu \rightarrow 0$. At large time these correlation functions are expected to flow towards some ‘universal’ functions manifesting a self-similar character. In other words

$$\left\langle \prod_{j=1}^n u(x_j, t) \right\rangle \simeq u^n(t) B_n \left(\frac{x_j}{l(t)} \right) \quad \text{for } t \text{ large} \quad (2)$$

with $l(t)$ and $u(t) \simeq \partial_t l(t)$ being the characteristic length and the characteristic velocity at time t . Note the order of the limits: first $\lim_{\nu \rightarrow 0}$ and then $\lim_{t \rightarrow \infty}$. These limits do not commute. See [7] for a recent discussion of the domain of validity of these asymptotic behaviours. Each asymptotic universal statistics, which are specified by their correlation functions $B_n(x_j/l(t))$, have their own basin of attraction.

If the initial statistics are Gaussian with zero mean, they are encoded in the initial velocity two-point function $\Gamma(x - y) = \langle u_0(x)u_0(y) \rangle$ which we assume is translation invariant. As is known since the work of Burgers, the large time behaviour depends crucially

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on whether $\mathcal{J} = \int dx \Gamma(x)$ vanishes or not. The case $\mathcal{J} \neq 0$ was the case first studied by Burgers himself [1]. In this case the large time behaviour is governed by a self-similar solution with characteristic length $l(t) \sim t^{2/3}$. The case $\mathcal{J} = 0$ was analysed by Kida in his important paper [2]. It is convenient to introduce the potential $\Phi(x, t)$ such that $u(x, t) = \partial_x \Phi(x, t)$. If $\mathcal{J} = 0$, we may assume that the initial potential is Gaussian with mean zero and the two-point function

$$G(x - y) = \langle \Phi_0(x) \Phi_0(y) \rangle. \quad (3)$$

Of course $\Gamma(x) = -\partial_x^2 G(x)$. Assuming *a priori* that the minima of the initial potential are independent, Kida showed in [2] that the large time behaviour has a characteristic length $l(t) \sim t^{1/2}$ up to a logarithmic correction. A more precise formulation of this statement, previously recorded in the physics literature (see [2, 5, 7] and references therein, and proved in [6], see also [7]), under a very mild hypothesis on $G(x)$, (e.g. for $G(x)$ being a smooth function decreasing rapidly), states that the following limit exists:

$$\lim_{\epsilon \rightarrow 0} |\log \epsilon|^{1/2} u \left(\frac{x}{\epsilon}, \frac{t}{\epsilon^2} \mid \log \epsilon \mid^{1/2} \right) \cong u_K(x, t). \quad (4)$$

Here and in the following, \cong means an equality in law, i.e. inside any correlation functions. The statistics of the limiting velocity field $u_K(x, t)$ was explicitly constructed in [2, 6]. $u_K(x, t)$ is self-similar with a diffusive scaling, $l_K(t) = t^{1/2} \Delta^{1/4}$ where $\Delta = G(0)$. Note that one of the hypothesis for having Kida's statistics at large time is that $0 < \Delta < \infty$, i.e. that the initial potential two-point function is regular at the origin.

Asymptotic behaviours of solutions with non-Gaussian initial data have also been considered in the literature. For example, [8] considers non-Gaussian initial data but whose asymptotic large time behaviours may still be described by self-similar solutions with a diffusive scaling $l(t) \sim \sqrt{t}$. See [7] for an extended commented list of references.

In this paper, we shall construct other self-similar solutions of decaying Burgers turbulence. Although these solutions differ from Kida's statistics, they share in common their construction from Poisson point processes. These solutions may be relevant in the large time behaviour of systems whose initial correlation functions are singular at coinciding points.

In section 2, we introduce a few basic facts concerning the Burgers equation and we describe some universal features of fields localized on shocks. In section 3, we construct the self-similar statistics and we prove that it is effectively a solution of the turbulent problem. We also give some more detailed information on a particular case. Comments and speculations are gathered in section 4.

2. Basic facts about the Burgers equation

In order to fix notations, we recall few elementary facts concerning the Burgers equation [1, 2]. As is well known, the equation is solved by implementing the Cole–Hopf transformation which maps it to the heat equation. This works as follows. Let $Z(x, t) = \exp[-\frac{1}{2v} \Phi(x, t)]$ where $u(x, t) = \partial_x \Phi(x, t)$. Equation (1) for u is mapped into the heat equation for Z :

$$[\partial_t - v \partial_x^2] Z(x, t) = 0.$$

Thus, given the initial condition $u(x, t = 0) \equiv u_0(x)$, the velocity field at a later time t is recovered from the potential $\Phi(x, t)$ given by the relation

$$\exp \left[-\frac{1}{2v} \Phi(x, t) \right] = \int \frac{dy}{\sqrt{4\pi vt}} \exp \left[-\frac{1}{2v} \left(\Phi_0(y) + \frac{(x-y)^2}{2t} \right) \right] \quad (5)$$

with $\Phi_0(x)$ standing for the initial potential such that $u_0(x) = \partial_x \Phi_0(x)$. The invicid Burgers equation corresponds to the limit $\nu \rightarrow 0$. The solution is then given by solving a minimalization problem:

$$u(x, t) = \partial_x \Phi(x, t) \quad \text{with} \quad \Phi(x, t) = \min_y \left(\Phi_0(y) + \frac{(x-y)^2}{2t} \right). \quad (6)$$

Outside shocks the minimum is reached for only one value y_* of y , the solution of the equation $u_0(y_*)t = x - y_*$. The velocity is $u(x, t) = \frac{x-y_*}{t} = u_0(y_*)$. It is effectively a local solution of the invicid Burgers equation since, by the minimum condition defining y_* , we have $u(x, t) = u_0(x - tu(x, t))$. A simple geometrical construction of the solution (6) is described in [1, 2]. For large t , y_* coincides approximately with one of the local minima of $\Phi_0(y)$ and it practically does not change under small variations of x so that, inbetween the shocks, the velocity is approximately linear with the slope $\frac{1}{t}$.

Shocks appear when the minimum is reached for two values y_1 and y_2 of y . Let $\Phi_{1,2} = \Phi_0(y_{1,2})$ be the value of the initial potential at these points. Then equation (5) allows one to determine the velocity profile $u_s(x, t)$ around and inside the shocks at finite value of the viscosity ν by expressing $\exp[-\frac{1}{2\nu}\Phi_s(x, t)]$ as the sum of contributions from the two minima. One obtains

$$u_s(x, t) = \frac{1}{t} \left(x - \frac{1}{2}(y_1 + y_2) \right) - \frac{\mu_s}{2t} \tanh \left(\frac{\mu_s}{4\nu t} \left(x - \xi_s t - \frac{1}{2}(y_1 + y_2) \right) \right) \quad (7)$$

where $\mu_s = y_1 - y_2 > 0$ and $\xi_s = \frac{\Phi_1 - \Phi_2}{y_1 - y_2}$. In the invicid limit $\nu \rightarrow 0$, this becomes

$$u_s(x, t)|_{\nu=0} = \xi_s + \frac{x - x_s(t)}{t} - \frac{\mu_s}{2t} (\theta(x - x_s(t)) - \theta(x_s(t) - x)) \quad (8)$$

where $x_s(t) = \xi_s t + \frac{1}{2}(y_1 + y_2)$ is the time t position of the shock which moves with the velocity ξ_s and follows a Lagrangian trajectory. $\theta(x)$ is the step function. The values of the velocity on the two sides of the shock are

$$u_s^\pm = u_s(x_s \pm 0) = \xi_s \mp \frac{\mu_s}{2t} \quad (9)$$

so that $\frac{\mu_s}{t}$ is the amplitude of the shock.

The presence of shocks is at the origin of universal features which are independent of the details of the statistics. They may be analysed by looking at fields localized on the shocks. By definition, these fields may be represented for any realization as

$$\mathcal{O}_g(x, t) = \sum_{\text{shocks}} g(\xi_s, \mu_s) \delta(x - x_s(t)) \quad (10)$$

where the sum is over the shocks with $x_s(t)$ denoting the position of the shock, ξ_s its velocity and $\frac{\mu_s}{t}$ its amplitude. These fields are labeled by functions of ξ_s and μ_s . By using the velocity profile (7) inside and around the shocks, we may map fields defined in terms of the velocity $u(x, t)$ into the shock representation. For example, the shifted derivative of the velocity field $(\partial_x u(x, t) - \frac{1}{t})$ is for large t localized on the shocks since away from shocks, $u(x, t) = \frac{x-y_*}{t}$ with y_* almost independent of x . More generally, the generating function $(\partial_x - \frac{\lambda}{t}) \exp[\lambda u(x, t)]$ also becomes localized on the shocks for large t . Using the velocity profiles (7) or directly (8), one finds its shock representation:

$$\left(\partial_x - \frac{\lambda}{t} \right) e^{\lambda u(x, t)} = -2 \sum_s e^{\lambda \xi_s} \sinh \left(\frac{\lambda \mu_s}{2t} \right) \delta(x - x_s(t)). \quad (11)$$

Note that this differs from the products $\lambda(\partial_x u(x, t) - \frac{1}{t})e^{\lambda u(x \pm 0, t)}$ which are evaluated using the fact that the velocity on the two sides of a shock are $u_s^\pm = \xi_s \mp \frac{\mu_s}{2t}$:

$$\lambda \left(\partial_x u(x, t) - \frac{1}{t} \right) e^{\lambda u(x \pm 0, t)} = -\lambda \sum_s \frac{\mu_s}{t} \exp \left[\lambda \left(\xi_s \mp \frac{\mu_s}{2t} \right) \right] \delta(x - x_s(t)). \quad (12)$$

Another example of a field localized on shocks is provided by the dissipation field $\epsilon(x, t)$ defined by $\epsilon(x, t) = \lim_{\nu \rightarrow 0} \nu (\partial_x u)^2$. As is well known, $\epsilon(x)$, which is naively zero due to the prefactor ν in its definition, is actually a non-trivial field since $(\partial_x u)^2$ is singular in the invicid limit. Integrating $\nu (\partial_x u_s)^2$ around the shock one obtains in the limit $\nu \rightarrow 0$ the contribution $\frac{1}{12} (\frac{\mu_s}{t})^3$. Hence the shock representation of $\epsilon(x)$ is:

$$\epsilon(x) = \frac{1}{12} \sum_s \left(\frac{\mu_s}{t} \right)^3 \delta(x - x_s(t)).$$

More generally one finds, using again the velocity profile (7), the shock representation of the generating function $\epsilon(x, t)e^{\lambda u(x, t)}$. Namely

$$\begin{aligned} \epsilon_\lambda(x, t) &\equiv \epsilon(x, t)e^{\lambda u(x, t)} \\ &= 2\lambda^{-3} \sum_s e^{\lambda \xi_s} \left(\frac{\lambda \mu_s}{2t} \cosh \left(\frac{\lambda \mu_s}{2t} \right) - \sinh \left(\frac{\lambda \mu_s}{2t} \right) \right) \delta(x - x_s(t)). \end{aligned} \quad (13)$$

Note that for $\nu \neq 0$, the Burgers equation (1) implies that

$$0 = \left(\partial_t + \lambda \partial_\lambda \frac{1}{\lambda} \partial_x - \lambda \nu (\partial_x^2 u) \right) e^{\lambda u} = \left(\partial_t + \lambda \partial_\lambda \frac{1}{\lambda} \partial_x + \lambda^2 \nu (\partial_x u)^2 - \nu \partial_x^2 \right) e^{\lambda u}. \quad (14)$$

Since $\partial_x^2 e^{\lambda u}$ has a (distributional) limit when $\nu \rightarrow 0$, we may expect that at $\nu = 0$

$$\left(\partial_t + \lambda \partial_\lambda \frac{1}{\lambda} \partial_x \right) e^{\lambda u} + \lambda^2 \epsilon_\lambda = 0 \quad (15)$$

encoding the invicid version of the Burgers equation. Indeed, equation (15) may be verified directly at $\nu = 0$ by computing $(\partial_t + \lambda \partial_\lambda \frac{1}{\lambda} \partial_x) e^{\lambda u}$ with the use of the limiting profile (8).

Comparison of equation (13) with equations (11) and (12) yields an alternative representation of the dissipation field in terms of the velocity field:

$$\begin{aligned} \epsilon_\lambda(x, t) &= \frac{1}{2} \lambda^{-2} (\partial_x u(x, t)) (2e^{\lambda u(x, t)} - e^{\lambda u(x+0, t)} - e^{\lambda u(x-0, t)}) \\ &= \frac{1}{2} \lambda^{-2} (e^{\lambda u(x, t)}) (2\partial_x u(x, t) - \partial_x u(x+0, t) - \partial_x u(x-0, t)). \end{aligned} \quad (16)$$

Equation (16) is an extension of the well known formula

$$\epsilon(x) = \frac{1}{12} \lim_{l \rightarrow 0} \partial_l [u(x) - u(x+l)]^3.$$

As expected and manifested in equation (16), the dissipation field is located on the discontinuity of the derivative of the velocity field. Equation (16) does not coincide with the operator product expansion suggested in [9] for the forced Burgers turbulence and expressing ϵ_λ as a combination of $e^{\lambda u}$ and $\partial_x e^{\lambda u}$.

Fields localized on shocks form a closed algebra. When shocks are diluted, these operators are expected to satisfy a simple operator product expansion:

$$\mathcal{O}_f(x, t) \cdot \mathcal{O}_g(y, t) = \delta(x - y) \mathcal{O}_{fg}(x, t) + \text{regular}.$$

The contact term $\delta(x - y)$ in this operator product expansion arises from the coinciding shocks in the double sum representing the product operator. As an application, let us

present an argument showing that the structure functions scale linearly in x . Indeed, using the representation (11) for the operator $(\partial_x - \lambda)e^{\lambda u(x,t)}$ at $t = 1$, one finds

$$(\partial_x - \lambda)^2 \langle e^{\lambda(u(x)-u(0))} \rangle = \delta(x) \langle \mathcal{O}_\varphi \rangle + \text{regular}$$

with $\mathcal{O}_\varphi(x) = 2 \sum_s (\cosh(\lambda \mu_s) - 1) \delta(x - x_s)$. By integrating, this implies

$$\langle e^{\lambda(u(x)-u(0))} \rangle = 1 + a(\lambda)x + \frac{\langle \mathcal{O}_\varphi \rangle}{2} |x| + o(|x|) \tag{17}$$

with $a(\lambda) = -a(-\lambda)$. Equation (17) implies that, at short distances, $\langle (u(x) - u(0))^n \rangle$ is proportional to $|x|$ for even positive n :

$$\langle (u(x) - u(0))^n \rangle = \langle \mathcal{O}_{\mu^n} \rangle |x| + o(|x|)$$

and it is consistent with the behaviour proportional to x for odd $n > 1$, as the one holding for the three-point function. In words, the anomalous scalings of the structure functions in Burgers turbulence are a simple echo of the shocks. They are universal (at least when shocks are diluted): only the amplitudes are statistics dependent. Some of these representations and formal manipulations also apply to the forced Burgers equation.

3. Self-similar solutions

Self-similar behaviour such as in equation (2) will be true at any time, i.e. not only asymptotically, if the initial correlation functions scale. Indeed, from the explicit solution (6) it immediately follows that (for the Gaussian initial data)

$$G_0(sx) = s^{2h} G_0(x) \implies s^{1-h} u(sx, s^{2-h}t) \cong u(x, t). \tag{18}$$

h is the dimension of the initial potential. Such scaling behaviour corresponds to a characteristic length $l(t) \sim t^{\frac{1}{2-h}}$. For this length to grow with time we must have $h < 2$. Shocks are expected to be dense for $1 < h < 2$ and diluted for $h < 1$ [3, 4].

Demanding a self-similar behaviour for the correlation functions imposes constraints on the correlation functions of the velocity field:

$$\left[(2-h)t \partial_t + \sum_j (x_j \partial_{x_j} + (1-h)\lambda_j \partial_{\lambda_j}) \right] \left\langle \prod_k e^{\lambda_k u(x_k, t)} \right\rangle = 0. \tag{19}$$

One may construct self-similar solutions with scaling dimensions h for $0 \geq h > -1$ by generalizing the representation of Kida's asymptotic solution [2]. By construction, the velocity $u_h(x, t)$ has the following form:

$$u_h(x, t) = \partial_x \Phi_h(x, t) \quad \text{with} \quad \Phi_h(x, t) = \min_j \left(\phi_j + \frac{(x - y_j)^2}{2t} \right) \tag{20}$$

where $(\phi_j, y_j)_{j \in \mathbb{Z}}$ is a Poisson point process with intensity $f_h(\phi) d\phi dy$. Recall that this means that the probability to find a point of this process in an infinitesimal cell centered at (ϕ, y) is $f_h(\phi, y) d\phi dy$ and that such elementary events are independent. To assure the translation invariance, f_h will depend only on ϕ . For any given realization, the velocity field (20) has an exact sawtooth profile with slope $\frac{1}{t}$. In this ansatz all shocks are created at time $t = 0$. The later time evolution is then governed by the shock collisions: the largest eating the smallest.

Let us first show that equation (20) is preserved by the evolution specified by the invicid Burgers equation. At a time $t' = t + \tau > t$, the velocity field $u_h(x, t')$ is given by

$$u_h(x, t + \tau) = \partial_x \min_y \left(\Phi_h(y, t) + \frac{(x - y)^2}{2\tau} \right).$$

Inserting the expression (20) for $\Phi(y, t)$ and commuting the minimization over y and over j 's, we get:

$$\begin{aligned} u_h(x, t + \tau) &= \partial_x \min_j \left(\phi_j + \min_y \left(\frac{(y - y_j)^2}{2t} + \frac{(x - y)^2}{2\tau} \right) \right) \\ &= \partial_x \min_j \left(\phi_j + \frac{(x - y_j)^2}{2(t + \tau)} \right). \end{aligned}$$

Next we determine the intensity $f_h(\phi) d\phi dy$ such that the solution (20) is self-similar with scaling dimension h , i.e. $s^{1-h} u_h(sx, s^{2-h}t) \cong u(x, t)$. Let us spell out this condition for the potential $\Phi_h(x, t)$. By definition,

$$\begin{aligned} s^{-h} \Phi_h(sx, s^{2-h}t) &= \min_j \left(s^{-h} \phi_j + \frac{(x - s^{-1}y_j)^2}{2t} \right) \\ &= \min_j \left(\hat{\phi}_j + \frac{(x - \hat{y}_j)^2}{2t} \right) \end{aligned}$$

where $\hat{\phi}_j = s^{-h} \phi_j$ and $\hat{y}_j = s^{-1} y_j$. Since $u_h(x, t) = \partial_x \Phi_h(x, t)$, demanding self-similarity amounts to require that $s^{-h} \Phi_h(sx, s^{2-h}t) \cong \Phi_h(x, t) - C_s$ with C_s a constant. This equality will be true in law if the intensity is such that: $f_h(\phi) d\phi dy = f_h(\hat{\phi} + C_s) d\hat{\phi} d\hat{y}$. Up to an irrelevant translation of ϕ the solutions of ϕ this equation are:

$$f_h(\phi) = \begin{cases} \text{constant } \phi^{-\left(\frac{1+h}{h}\right)} & \text{for } \phi \geq 0 \\ 0 & \text{for } \phi \leq 0 \end{cases} \quad \text{with } -1 < h < 0 \quad (21)$$

$$f_0(\phi) = \exp[\text{constant } \phi] \quad \text{with } h = 0. \quad (22)$$

Thus we have shown that the representation (20) of the velocity field in terms of the Poisson process (ϕ_j, y_j) with intensity (21) is (i) self-similar and (ii) preserved by the evolution. In particular the relations such as equations (19) will be satisfied. Moreover the invicid form (15) of the Burgers equation for each realization implies the Hopf equations for the correlators:

$$\left[\partial_t + \sum_j \lambda_j \partial_{\lambda_j} \frac{1}{\lambda_j} \partial_{x_j} \right] \left\langle \prod_k e^{\lambda_k u(x_k, t)} \right\rangle + \sum_j \lambda_j^2 \left\langle \epsilon_{\lambda_j} \prod_{k \neq j} e^{\lambda_k u(x_k, t)} \right\rangle = 0 \quad (23)$$

The time derivative may be eliminated from both equations leading to the fixed-time version of the Hopf equations. The case $h = 0$ corresponds to Kida's asymptotic solution. As pointed out in [10] it generalizes the Gumbel class of extreme statistics. The case $h \neq 0$ should, correspondingly, generalize the Weibull class of extreme statistics.

Let us describe in more detail the case $h = -\frac{1}{2}$. It corresponds to initial potential correlation functions homogeneous of degree -1 , e.g. like the Dirac delta function. We denote by $\Phi^*(x, t)$ and $u^*(x, t)$ the corresponding potential and velocity field. The statistics of the Poisson point process (ϕ_j, y_j) in the representation (20) is specified by the intensity

$$f^*(\phi) = \begin{cases} D^{-1} \phi d\phi dy & \text{for } \phi \geq 0 \\ 0 & \text{for } \phi \leq 0. \end{cases} \quad (24)$$

D is a constant with dimension $(\text{length})^5 \times (\text{time})^{-2}$. Recall that the velocity field $u^*(x, t)$ is such that $u^*(x, t) \cong t^{-3/5} u^*(xt^{-2/5}, 1)$. In other words, the characteristic length at time t is $l^*(t) = D^{1/5} t^{2/5}$.

The one-point function of the potential scales as $t^{-2/5}$ and thus diverges at $t = 0$ (it does not contribute to the one-point function of the velocity which vanishes). The two-point function $G^*(x, t)$ of $\Phi^*(x, t)$ satisfies $G^*(x, t) = t^{-2/5} G^*(xt^{-2/5}, 1)$. Since, as we shall

see, $G^*(x, 1)$ is smooth, regular at the origin and decreasing exponentially at infinity, at zero time $G(x, t)$ becomes proportional to the Dirac delta function,

$$\lim_{t \rightarrow 0} G^*(x, t) = \text{constant } \delta(x).$$

Once the velocity field has been parametrized in terms of Poisson processes as in equation (20), it is easy to compute any correlation functions. For example, the one-point generating function is

$$\langle e^{\lambda u^*(x,t)} \rangle = \int d\phi dy \mathcal{P}_x(\phi, y) e^{\lambda(x-y)/t}$$

with $\mathcal{P}_x(\phi, y)$ denoting the probability (density) of $(\phi, y) = (\phi_{j^*}, y_{j^*})$ for j^* minimizing $(\phi_j + \frac{(x-y_j)^2}{2t})$. To compute this probability, imagine dividing the (ϕ, y) -plane into elementary cells of size $d\phi dy$ with the probability for a point (ϕ_j, y_j) to be in the cell centred at (ϕ, y) equal to $f(\phi) d\phi dy$. Thus, the probability $\mathcal{P}_x(\phi, y)$ is equal to the product of the probability for the elementary cell centred around (ϕ, y) to be occupied by a point of the process times the probability for the other cells around (ϕ', y') s.t. $\phi' + \frac{(x-y')^2}{2t} < \phi + \frac{(x-y)^2}{2t}$ to be empty (if this condition is violated, the cell around (ϕ', y') may be either occupied or empty). Hence

$$\mathcal{P}_x(\phi, y) d\phi dy = f(\phi) d\phi dy \prod_{d\phi' dy'} (1 - \chi(\phi', y'; \phi, y) f(\phi') d\phi' dy')$$

where $\chi(\phi', y'; \phi, y)$ is the characteristic function of the constraint $\phi' + \frac{(x-y')^2}{2t} < \phi + \frac{(x-y)^2}{2t}$. Taking the continuum limit of infinitesimal cells and approximating the product over the cells around (ϕ', y') by the exponential of a sum leads to the result:

$$\begin{aligned} \langle e^{\lambda u^*(x,1)} \rangle &= \int d\phi dy e^{-\lambda y} f\left(\phi - \frac{y^2}{2}\right) \exp\left[-\int_{-\infty}^{\phi - \frac{y^2}{2}} dz \int_{-\infty}^{\phi - \frac{z^2}{2}} d\phi' f(\phi')\right] \\ &= \frac{2}{|\lambda|^{5/2}} \int_0^\infty dX X (X \cosh X - \sinh X) \exp\left[-\frac{2}{15} (X/|\lambda|)^5\right]. \end{aligned} \quad (25)$$

We have set $t = 1$ and $D = 1$. The dependence on these parameters is restored by replacing λ by $\lambda D^{1/5} t^{-3/5}$. One may check directly that the above expression for the one-point function satisfies the identities (19) and (23). $\langle e^{\lambda u^*(x,1)} \rangle$ is regular around $\lambda = 0$ and its behaviour at infinity is:

$$\langle e^{\lambda u^*(x,1)} \rangle \simeq \text{constant} |\lambda|^{-15/8} \exp[+\text{constant} |\lambda|^{5/4}] \quad \text{for } \lambda \rightarrow \infty. \quad (26)$$

This has to be compared with Kida's statistics for which $\langle e^{\lambda u^*(x,1)} \rangle = \exp[+\text{constant } \lambda^2]$. The two-point functions can be computed similarly. For $x > 0$,

$$\langle e^{\lambda(u^*(x) - u^*(-x))} \rangle = e^{2\lambda x} \int_{\phi > 0} d\phi dy [\phi + 2x J_\lambda(\phi; y; x) J_\lambda(\phi; -y; x)] e^{-I_0(\phi; y; x) - I_0(\phi; -y; x)} \quad (27)$$

with

$$\begin{aligned} J_\lambda(\phi; y; x) &= \int_D dz (\phi + \frac{1}{2}(x+y)^2 - \frac{1}{2}z^2) e^{\lambda(z-x)} \\ I_0(\phi; y; x) &= \frac{1}{2} \int_D dz (\phi + \frac{1}{2}(x+y)^2 - \frac{1}{2}z^2)^2 \end{aligned}$$

where the domain of integration in both cases is $\mathcal{D} = \{z \mid z \leq y + x; z^2 \leq 2\phi + (y + x)^2\}$. As it should be, equation (27) gives structure functions which scale linearly in x at small distances:

$$\langle (u(x) - u(0))^n \rangle \propto x + \dots$$

for any positive odd integer n . For n even the structure functions are linear in $|x|$. At large distances, the structure functions $\langle (u(x) - u(0))^n \rangle$ decrease faster than $B_n |x|^{\sigma_n} \exp(-A_n |x|^5)$ with A_n, B_n and σ_n some n -dependent constants. Equation (27) may be used to check that for $t \neq 0$ the two-point function $G(x, t)$ is smooth, fast decreasing at infinity and regular at the origin. There are no difficulties, but not much motivation, to compute in the same way the higher-point correlation functions.

4. Comments and speculations

We have constructed self-similar solutions of the decaying Burgers turbulence. It is natural to wonder if such statistics effectively describe the long time behaviour of systems with smooth random initial data. One may construct such examples in a tautological way by taking as initial data the potential $\Phi_0^\eta(x)$ obtained from the self-similar ansatz (20) at small but non-zero time: $\Phi_0^\eta(x) = \Phi_h(x, \eta)$ with $\eta \neq 0$. By construction, it defines a smooth initial statistics which will have a large time asymptotics given by $\Phi_h(x, t)$. Of course, this initial statistics is not Gaussian. This shows, however, that the basin of attraction of the self-similar solution (20) is not totally empty.

In the case $h = -\frac{1}{2}$ the two-point function of $\Phi_0^\eta(x)$ tends to the Dirac delta function $\delta(x - y)$ when $\eta \rightarrow 0$. So the initial statistics $\Phi_0^\eta(x)$ may be thought of as a way to regularize an initial potential with $\delta(x - y)$ two-point correlation function. Note that if we replace $\delta(x - y)$ by a smooth cut-off dependent function $\Delta(x - y)$, the values at the origin $\Delta(0)$ diverge with the cut-off and Kida's asymptotic regime disappears in the limit since its characteristic length diverges.

The question is then whether there exist analogues of equation (4) but with different respective scaling between x and t corresponding to non-zero values of h . For example one may inquire about the existence of the limit

$$\Phi^{**}(x, t) \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{1/2}} \Phi\left(\frac{x}{\epsilon}, \frac{t}{\epsilon^{5/2}}\right). \quad (28)$$

Naively, it corresponds to a limiting initial potential with $\delta(x - y)$ correlation function. Indeed, upon assuming that \min_y and $\lim_{\epsilon \rightarrow 0}$ commute, it follows from equation (6) that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{1/2}} \Phi\left(\frac{x}{\epsilon}, \frac{t}{\epsilon^{5/2}}\right) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{1/2}} \min_y \left(\Phi(y) + \epsilon^{5/2} \frac{(x\epsilon^{-1} - y)^2}{2t} \right) \\ &= \min_y \left(\Phi_0^{**}(y) + \frac{(x - y)^2}{2t} \right) \end{aligned} \quad (29)$$

where $\Phi_0^{**}(x) \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{1/2}} \Phi\left(\frac{x}{\epsilon}\right)$ is the rescaled initial potential with the two-point function $G^{**}(x) \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} G\left(\frac{x}{\epsilon}\right) = \bar{D} \delta(x)$ with $\bar{D} = \int dx G(x)$. It has dimension $h = -\frac{1}{2}$. Such a limit (28) would correspond to an intermediate regime with characteristic length $l^*(t) \sim t^{2/5}$ smaller than Kida's length $l_K(t) \sim t^{1/2}$. The problem is, however, that the realizations of the white noise Φ_0^{**} are distributional and that the last expression in (29) is ill defined. Clearly a finer analysis is required to decipher cases in which a limit of the type (28) leads to a self-similar asymptotic distribution with $h = -\frac{1}{2}$ as the one constructed above. In fact,

we expect the latter to appear if the initial potentials have values uniformly bounded below, in analogy with the extreme statistics problem.

Identical constructions, arguments and speculations could be conducted in higher dimensions. For example, in dimension d the delta function has dimension $h = -\frac{d}{2}$, and there exists a self-similar solution with this scaling dimension. It corresponds to the characteristic length $l(t) \sim t^\alpha$ with $\alpha = \frac{2}{d+4}$.

Finally, we feel that it could be worthwhile to adapt the renormalization group techniques to analyse this type of large time behaviour.

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